QUATERNION H-TYPE GROUP AND DIFFERENTIAL OPERATOR Δ_λ

DER-CHEN CHANG, IRINA MARKINA

ABSTRACT. We study the relations between the quaternion H-type group and the boundary of the unit ball on two dimensional quaternionic space. The orthogonal projection of the space of square integrable functions defined on quaternion H-type group into its subspace of boundary values of q-holomorphic functions is consider. The precise form of Cauchy-Szego kernel and the orthogonal projection operator is obtained. The fundamental solution for the operator Δ_λ is found.

Dedicated to Professor Qikeng Lu on his 80th birthday

1. INTRODUCTION

The real number system is extended to the complex then to the quaternion systems of numbers and further finds its the most exciting generalization incorporated the geometric concept of the direction, the so-called Clifford algebras, which Clifford himself called "geometric algebras", [17, 18], see also [13] for Clifford analysis and numerous references. The division algebras, which are only real, complex, quaternionic and octonionic numbers give the origin to homogeneous groups satisfying J^2 condition [5]. The simplest non-commutative example of them is the Heisenberg group closely related to the complex number system and that found its numerous applications in physics, quantum mechanics, differential geometry (see, for instance [2, 15, 16]). The following more complicate example is an quaternion analogue of the Heisenberg group that has at least four dimensional horizontal distribution and three dimensional center [4]. We call this group quaternion H-type group due to [14].

The quaternion H-type group $\mathcal{Q}$ can be realized as a boundary of the unit ball on the two dimensional quaternionic space $\mathbb{H}^2$. The Siegel upper half space $\mathcal{U}_1 \in \mathbb{H}^2$ is q-holomorphically equivalent to the unit ball in $\mathbb{H}^2$. The group $\mathcal{Q}$ arise as the group of translations of $\mathcal{U}_1$. This leads to its identification with the boundary $\partial\mathcal{U}_1$. We give the precise formulas of the action. Because of this identification and by the use of various symmetries of $\mathcal{U}_1$ the Cauchy-Szegö projector is realized as a convolution operator on the group $\mathcal{Q}$ with explicitly given singular kernel. The analogue for the group $\mathcal{Q}$ of the Laplace operator is the so called Δ_λ operator that is expressed as a sum of the square of vector fields forming the frame of the horizontal distribution plus the λ times their commutators. We find the kernel of the operator Δ_λ.

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2. Quaternion H-type group and the Siegel upper half space.

We remember shortly the definitions of quaternion numbers. Let i_1, i_2, i_3 be three imaginary units such that
$$i_1^2 = i_2^2 = i_3^2 = i_1 i_2 i_3 = -1.$$
The multiplication between the imaginary units is given in Table 1. Any quaternion q can

Table 1. Multiplication of imaginary units

	i_1	i_2	i_3
i_1	-1	i_3	$-i_2$
i_2	$-i_3$	-1	i_1
i_3	i_2	$-i_1$	-1

be written in the algebraic form as $q = t + ai_1 + bi_2 + ci_3$, where t, a, b, c are real numbers. The number t is called the *real part* and denoted by $t = \mathrm{Re}\, q$. The vector $\mathbf{u} = (a, b, c)$ is the *imaginary part* of q. We use the notations
$$a = \mathrm{Im}_1 q, \quad b = \mathrm{Im}_2 q, \quad c = \mathrm{Im}_3 q, \quad \text{and} \quad \mathrm{Im}\, q = \mathbf{u} = (a, b, c).$$
Similarly to complex numbers, vectors, and matrices, the addition of two quaternions is equivalent to summing up the elements. Set $q = t + \mathbf{u}$, and $h = s + xi_1 + yi_2 + zi_3 = s + \mathbf{v}$. Then
$$q + h = (t + s) + (\mathbf{u} + \mathbf{v}) = (t + s) + (a + x)i_1 + (b + y)i_2 + (c + z)i_3.$$
Addition satisfies all the commutation and association rules of real and complex numbers. The quaternion multiplication (the Grassmanian product) is defined by
$$qh = (ts - \mathbf{u} \cdot \mathbf{v}) + (t\mathbf{v} + s\mathbf{u} + \mathbf{u} \times \mathbf{v}),$$
where $\mathbf{u} \cdot \mathbf{v}$ is the scalar product and $\mathbf{u} \times \mathbf{v}$ is the vector product of $\mathbf{u}$ and $\mathbf{v}$. The multiplication is not commutative because of the non-commutative vector product. The conjugate $\bar{q}$ to q is defined in a similar way as for the complex numbers: $\bar{q} = t - ai_1 - bi_2 - ci_3$. Then the modulus $|q|$ of q is given by $|q|^2 = q\bar{q} = t^2 + a^2 + b^2 + c^2$. The scalar product between q and h is

(2.1) $$\langle q, h \rangle = \mathrm{Re}(q\bar{h}) = ts + ax + by + cz.$$

We also have
$$\overline{qh} = \bar{h}\bar{q}, \quad |qh| = |q||h|, \quad q^{-1} = \frac{\bar{q}}{|q|^2}.$$

The imaginary units have the representation by (4×4) real matrices:
$$\mathbf{i}_1 = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \quad \mathbf{i}_2 = \begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}, \quad \mathbf{i}_3 = \begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}.$$
Then a quaternion can be written in the matrix form
$$Q = \begin{bmatrix} t & -a & -b & -c \\ a & t & -c & b \\ b & c & t & -a \\ c & -b & a & t \end{bmatrix} = t\mathbf{U} + a\mathbf{i}_1 + b\mathbf{i}_2 + c\mathbf{i}_3,$$
where $\mathbf{U}$ is the unit (4×4) matrix. Notice that

1. $\det Q = |q|^4$,

2. $Q^T = -Q$ represents the conjugate quaternion $\bar{q}$,
3. $Q^{-1} = -\frac{1}{\sqrt{\det Q}}Q$,
4. Q^{-1} represents the inverse quaternion q^{-1}.

The nice description of algebraic and geometric properties of quaternion a reader can find in the original book of W. Hamilton [7].

Let $\mathbb{H}^2$ be a two dimensional vector space of pairs $h = (h_1, h_2)$ over the field of real numbers with the norm $\|h\|^2 = h_1^2 + h_2^2$. We describe the Siegel upper half space in $\mathbb{H}^2$ carrying out the counterpart with the two dimensional complex space $\mathbb{C}^2$.

Let D_1 denotes the unit ball in $\mathbb{C}^2$:

$$D_1 = \{(w_1, w_2) \in \mathbb{C}^2 : |w_1|^2 + |w_2|^2 < 1\}.$$

The set

$$U_1 = \{(z_1, z_2) \in \mathbb{C}^2 : \operatorname{Re} z_2 > |z_1|^2\}$$

is the Siegel half space. The Caley transformation

$$w_1 = \frac{2z_1}{1 + z_2}, \qquad w_2 = \frac{1 - z_2}{1 + z_2},$$

and its inverse

$$z_1 = \frac{w_1}{1 + w_2}, \qquad z_2 = \frac{1 - w_2}{1 + w_2},$$

show that the unit ball D_1 and Siegel half space U_1 are biholomorphically equivalent.

Now, take the unit ball B_1 in $\mathbb{H}^2$

$$B_1 = \{(h_1, h_2) \in \mathbb{H}^2 : |h_1|^2 + |h_2|^2 < 1\}$$

and a Siegel half space in $\mathbb{H}^2$

$$\mathcal{U}_1 = \{(q_1, q_2) \in \mathbb{H}^2 : \operatorname{Re} q_2 > |q_1|^2\}.$$

The Caley transformation, mapping the unit ball B_1 to the Siegel half space $\mathcal{U}_1$ and vice versa has the form:

$$h_1 = q_1\big(1 + (1 + q_2)^{-1}(1 - q_2)\big) = \frac{2q_1(1 + \bar{q}_2)}{|1 + q_2|^2}, \quad h_2 = (1 + q_2)^{-1}(1 - q_2) = \frac{(1 + \bar{q}_2)(1 - q_2)}{|1 + q_2|^2},$$

and the inverse transformation

$$q_1 = h_1(1 + h_2)^{-1} = \frac{h_1(1 + \bar{h}_2)}{|1 + h_2|^2}, \quad q_2 = (1 - h_2)(1 + h_2)^{-1} = \frac{(1 - h_2)(1 + \bar{h}_2)}{|1 + h_2|^2}.$$

Since the multiplication of quaternion is not commutative, it is possible to define another Caley's transformation, but the geometry will be the same. The boundary of $\mathcal{U}_1$ is

$$\partial\mathcal{U}_1 = \{(q_1, q_2) \in \mathbb{H}^2 : \operatorname{Re} q_2 = |q_1|^2\}.$$

We mention here three automorphisms of the domain $\mathcal{U}_1$, dilation, rotation and translation. Let $q = (q_1, q_2) \in \mathcal{U}_1$. For each positive number δ we define a *dilation* $\delta \circ q$ by

$$\delta \circ q = \delta \circ (q_1, q_2) = (\delta q_1, \delta^2 q_2).$$

The non-isotropy of the dilation comes from the definition of $\mathcal{U}_1$. For each unitary linear transformation $\mathcal{R}$ on $\mathbb{H}$ we define the *rotation* $\mathcal{R}(q)$ on $\mathcal{U}_1$ by

$$\mathcal{R}(q) = \mathcal{R}(q_1, q_2) = (\mathcal{R}(q_1), q_2).$$

Both, the dilation and rotation give q-holomorphic (the definition of q-holomorphic mapping see below) self mappings of $\mathcal{U}_1$ and extend to mappings on the boundary $\partial\mathcal{U}_1$. Before we

describe a translation on $\mathcal{U}_1$, we introduce the quaternionic H-type group denoted by $\mathcal{Q}$. This group consists of the set

$$\mathbb{H} \times \mathbb{R}^3 = \{[w, t] : \ w \in \mathbb{H}, t = (t_1, t_2, t_3) \in \mathbb{R}^3\}$$

with the multiplication law

(2.2) $[w, t_1, t_2, t_3] \cdot [\omega, s_1, s_2, s_3] = [w+\omega, t_1+s_1-2\operatorname{Im}_1 \bar\omega w, t_2+s_2-2\operatorname{Im}_2 \bar\omega w, t_3+s_3-2\operatorname{Im}_3 \bar\omega w].$

The law (2.2) makes $\mathbb{H} \times \mathbb{R}^3$ into Lie group with the neutral element $[0, 0]$ and the inverse element $[w, t]^{-1}$ given by $[w, t_1, t_2, t_3]^{-1} = [-w, -t_1, -t_2, -t_3]$.

To each element $[w, t]$ of $\mathcal{Q}$ we associate the following q-holomorphic affine self mapping of $\mathcal{U}_1$.

(2.3) $\qquad [w, t_1, t_2, t_3] : (q_1, q_2) \mapsto (q_1 + w, q_2 + |w|^2 + 2\bar w q_1 + i_1 t_1 + i_2 t_2 + i_3 t_3).$

This mapping preserve the following "height" function

(2.4) $\qquad\qquad\qquad\qquad r(q) = \operatorname{Re} q_2 - |q_1|^2.$

In fact, since $|q_1 + w|^2 = |q_1|^2 + |w|^2 + 2\operatorname{Re} \bar w q_1$, we obtain

$$\operatorname{Re}(q_2 + |w|^2 + 2\bar w q_1) - |q_1 + w|^2 = \operatorname{Re} q_2 - |q_1|^2.$$

Hence, the transformation (2.3) maps $\mathcal{U}_1$ to itself and preserves the boundary $\partial\mathcal{U}_1$.

The reader can check that the mapping (2.3) defines an action of the group $\mathcal{Q}$ on the space $\mathcal{U}_1$. If one composes the mappings (2.3), corresponding to elements $[w, t], [\omega, s] \in \mathcal{Q}$, the resulting transformation will correspond to the element $[w, t] \cdot [\omega, s]$. Thus, (2.3) gives us a realization of $\mathcal{Q}$ as a group of affine q-holomorphic bijections of $\mathcal{U}_1$. We can identify the elements of $\mathcal{U}_1$ with the boundary via its action on the origin

$$h(0) = [w, t] : (0, 0) \mapsto (w, |w|^2 + i_1 t_1 + i_2 t_2 + i_3 t_3),$$

where $h = [w, t]$. Thus

$$\mathcal{Q} \ni [w, t_1, t_2, t_3] \mapsto (w, |w|^2 + i_1 t_1 + i_2 t_2 + i_3 t_3) \in \partial\mathcal{U}_1.$$

We may use the following coordinates $(q_1, t, r) = (q_1, t_1, t_2, t_3, r)$ on $\mathcal{U}_1$:

$$\mathcal{U}_1 \ni (q_1, q_2) = (q_1, t_1, t_2, t_3, r),$$

where

$$t_1 = \operatorname{Im}_1 q_2, \quad t_2 = \operatorname{Im}_2 q_2, \quad t_3 = \operatorname{Im}_3 q_2, \quad r = r(q_1, q_2) = \operatorname{Re} q_2 - |q_1|^2.$$

If $\operatorname{Re} q_2 = |q_1|^2$ we get the coordinates on the boundary $\partial\mathcal{U}_1$ of the Siegel half space

$$\partial\mathcal{U}_1 \ni (q_1, q_2) = (q_1, t_1, t_2, t_3),$$

where t_m are as above and $r = r(q_1, q_2) = 0$.

3. Tangential Cauchy-Riemann-Fueter operators

Before going further, we collect here the necessary definitions concerning the quaternion calculus. Q-holomorphic functions on $\mathbb{H}$ were studied by Fueter and his collaborators [11, 12]. The reader can find the account of the theory of q-holomorphic functions in [6, 13, 20].

We continue exploit the analogy with the complex variables theory. Let M be a manifold of dimension $2n$ with the complex structure I. Such kind of manifolds is called *complex manifold*. Let $f : M \to \mathbb{C}$ be a differentiable function defined on the complex manifold M. Write $f = f_0 + i f_1$, where $f_0, f_1 : M \to \mathbb{R}$. Then f is called holomorphic, if

(3.1) $\qquad\qquad\qquad\qquad df_0 + I(df_1) = 0$

on M, where (3.1) is called the Cauchy-Riemann equation. Let, now, M be a manifold of dimention $4n$. A hypercomplex structure on M is a triple (I_1, I_2, I_3) on M, where I_k is a complex structure on M and $I_1 I_2 = I_3$. If M has hypercomplex structure, it is called the hypercomplex manifold. Let $f : M \to \mathbb{H}$ be a smooth function defined on a hypercomplex manifold M. Then $f = f_0 + i_1 f_1 + i_2 f_2 + i_3 f_3$, where $f_0, \ldots, f_3$ are smooth functions. We define a q-holomorphic function on M to be a smooth function $f : M \to \mathbb{H}$ for which

(3.2) $$df = df_0 + I_1(df_1) + I_2(df_2) + I_3(df_3) = 0.$$

Equation (3.2) is the natural quaternionic analogue of the Cauchy-Riemann equation (3.1) and is called the Cauchy-Riemann-Fueter equation.

The equation (3.2) can be written in terms of the partial derivatives with respect to the quaternionic variable. We introduce the differential operators

(3.3)
$$\frac{\partial_l f}{\partial \bar{q}} = \bar{\partial}_l f = \frac{1}{2}\left(\frac{\partial f}{\partial x_0} + \sum_{m=1}^{3} i_m \frac{\partial f}{\partial x_m}\right), \qquad \frac{\partial_r f}{\partial \bar{q}} = \bar{\partial}_r f = \frac{1}{2}\left(\frac{\partial f}{\partial x_0} + \sum_{m=1}^{3} \frac{\partial f}{\partial x_m} i_m\right),$$

$$\frac{\partial_l f}{\partial q} = \partial_l f = \frac{1}{2}\left(\frac{\partial f}{\partial x_0} - \sum_{m=1}^{3} i_m \frac{\partial f}{\partial x_m}\right), \qquad \frac{\partial_r f}{\partial q} = \partial_r f = \frac{1}{2}\left(\frac{\partial f}{\partial x_0} - \sum_{m=1}^{3} \frac{\partial f}{\partial x_m} i_m\right),$$

$$\Delta f = \frac{\partial^2 f}{\partial x_0^2} + \frac{\partial f^2}{\partial x_1^2} + \frac{\partial^2 f}{\partial x_2^2} + \frac{\partial^2 f}{\partial x_3^2}.$$

Note that $\bar{\partial}_l, \partial_l, \bar{\partial}_r$ and ∂_r all commute, and $\Delta = 4\partial_r \bar{\partial}_r = 4\partial_l \bar{\partial}_l$. Since the theories generated by ∂_l and ∂_r are symmetric, we shall use the left operator ∂_l and call the function simply q-holomorphic or regular. If $f : M \to \mathbb{H}$ a real-differentiable function, the Cauchy-Riemann-Fueter equation (3.2) can be written as $\bar{\partial}_l f = 0$ i. e.

$$\bar{\partial}_l f = \frac{\partial_l f}{\partial \bar{q}} = \frac{\partial f}{\partial (x_0)_k} + \sum_{m=1}^{3} i_m \frac{\partial f}{\partial (x_m)_k} = 0, \quad k = 1, \ldots, n.$$

We say that vector fields are *tangential Cauchy-Riemann-Fueter operators* if the following holds.

(i) Vector fields on $\partial \mathcal{U}^1$ arise by restricting (to $\partial \mathcal{U}^1$) of vector fields that, in coordinates (q_1, q_2) of $\mathbb{H}^2$, can be written in the form

(3.4) $$\alpha_1 \frac{\partial_l}{\partial \bar{q}_1} + \alpha_2 \frac{\partial_l}{\partial \bar{q}_2},$$

where the α_j are quaternion valued functions. This is the class of first-order differential operators that annihilate q-holomorphic functions.

(ii) Vector fields are tangential at $\partial \mathcal{U}^1$ in the sense that

$$\alpha_1 \frac{\partial_l r(q)}{\partial \bar{q}_1} + \alpha_2 \frac{\partial_l r(q)}{\partial \bar{q}_2} = 0,$$

wherever $r(q) = 0$. Here $r(q)$ is the defining function given by (2.4).

We consider the vector fields on the boundary $\partial \mathcal{U}_1$ of the Siegel half space with the coordinates $(q, t) = (x_0, x_1, x_2, x_3, t_1, t_2, t_3)$:

(3.5)
$$X_0(q, t) = \partial_{x_0} - 2x_1 \partial_{t_1} - 2x_2 \partial_{t_2} - 2x_3 \partial_{t_3},$$
$$X_1(q, t) = \partial_{x_1} + 2x_0 \partial_{t_1} - 2x_3 \partial_{t_2} + 2x_2 \partial_{t_3},$$
$$X_2(q, t) = \partial_{x_2} + 2x_3 \partial_{t_1} + 2x_0 \partial_{t_2} - 2x_1 \partial_{t_3},$$
$$X_3(q, t) = \partial_{x_3} - 2x_2 \partial_{t_1} + 2x_1 \partial_{t_2} + 2x_0 \partial_{t_3}.$$

They form a basis of the subbundle $\mathcal{T}(\partial\mathcal{U}_1)$ of the tangent bundle $T(\partial\mathcal{U}_1)$. In the same time the vector fields can be considered as a basis of Lie algebra, which is an infinitesimal representation of the group $\mathcal{Q}$. The commutators between the vector fields (3.5) is given in Table 2. Let us

TABLE 2. Commutators of the vector fields (3.5)

	X_0	X_1	X_2	X_3
X_0	0	$4\partial_{t_1}$	$4\partial_{t_2}$	$4\partial_{t_3}$
X_1	$-4\partial_{t_1}$	0	$-4\partial_{t_3}$	$4\partial_{t_2}$
X_2	$-4\partial_{t_2}$	$4\partial_{t_3}$	0	$-4\partial_{t_1}$
X_3	$-4\partial_{t_3}$	$-4\partial_{t_2}$	$4\partial_{t_1}$	0

verify, that the vector fields (3.5) are tangential Cauchy-Riemann-Fueter operators. To show this, we define the quaternion vector fields

$$
(3.6) \quad
\begin{aligned}
\bar{H}(w,t) &= \frac{1}{2}\big(X_0 + i_1 X_1 + i_2 X_2 + i_3 X_3\big) = \frac{\partial_l}{\partial \bar{w}} + w i_1 \partial_{t_1} + w i_2 \partial_{t_2} + w i_3 \partial_{t_3}, \\
H(w,t) &= \frac{1}{2}\big(X_0 - i_1 X_1 - i_2 X_2 - i_3 X_3\big) = \frac{\partial_l}{\partial w} - i_1 \bar{w} \partial_{t_1} - i_2 \bar{w} \partial_{t_2} - i_3 \bar{w} \partial_{t_3},
\end{aligned}
$$

where the terms $w i_k$ and $i_k \bar{w}$, $k = 1, 2, 3$, are the quaternion product. Then the commutative relation is $[\bar{H}, H] = 2 \sum_{k=1}^{3} i_k \partial_{t_k}$.

Lemma 3.1. *Using the identification of $\mathcal{Q}$ with $\partial\mathcal{U}_1$, the vector field $\bar{H}$ is tangential Cauchy-Riemann-Fueter operator on $\partial\mathcal{U}_1$.*

Proof. We remember the quaternion coordinates (w, t, r) on $\mathcal{U}_1$: for $(q_1, q_2) \in \mathcal{U}_1$

$$
w = q_1, \quad t_k = \text{Im}_k q_2, \ k = 1, 2, 3, \quad r(q_1, q_2) = \text{Re}\, q_2 - |q_1|^2.
$$

We use the left multiplication chain rule

$$
\frac{\partial_l}{\partial \bar{q}_m} = \frac{\partial_l \bar{w}}{\partial \bar{q}_m} \frac{\partial_l}{\partial \bar{w}} + \frac{\partial_l w}{\partial \bar{q}_m} \frac{\partial_l}{\partial w} + \sum_{k=1}^{3} \frac{\partial_l t_k}{\partial \bar{q}_m} \frac{\partial}{\partial t} + \frac{\partial_l r}{\partial \bar{q}_m} \frac{\partial}{\partial r}, \quad m = 1, 2.
$$

Calculating the derivatives

$$
\frac{\partial_l w}{\partial \bar{q}_m} = \frac{\partial_l \bar{w}}{\partial \bar{q}_2} = \frac{\partial_l t_k}{\partial \bar{q}_1} = 0, \quad \frac{\partial_l \bar{w}}{\partial \bar{q}_1} = 1, \quad \frac{\partial_l t_k}{\partial \bar{q}_2} = \frac{i_k}{2}, \quad \frac{\partial_l r}{\partial \bar{q}_1} = -q_1, \quad \frac{\partial_l r}{\partial \bar{q}_2} = \frac{1}{2},
$$

for $k = 1, 2, 3$, $m = 1, 2$, we conclude

$$
\frac{\partial_l}{\partial \bar{q}_1} = \frac{\partial_l}{\partial \bar{w}} - q_1 \frac{\partial_l}{\partial r}, \qquad \frac{\partial_l}{\partial \bar{q}_2} = \sum_{k=1}^{3} \frac{i_k}{2} \frac{\partial_l}{\partial \bar{t}_k} + \frac{1}{2} \frac{\partial_l}{\partial r}.
$$

Multiplying the second equation by $2q_1$ from the left and summing the derivative, we get

$$
\frac{\partial_l}{\partial \bar{w}} + \sum_{m=1}^{3} w i_m \partial_{t_m} = \frac{\partial_l}{\partial \bar{q}_1} + 2q_1 \frac{\partial_l}{\partial \bar{q}_2}.
$$

Notice that $\frac{\partial_l r}{\partial \bar{q}_1} + 2q_1 \frac{\partial_l r}{\partial \bar{q}_2} = 0$. This proves that the operator $\frac{\partial_l}{\partial \bar{q}_1} + 2q_1 \frac{\partial_l}{\partial \bar{q}_2}$ is tangential Cauchy-Riemann-Fueter operator. $\qquad\square$

4. THE CAUCHY-SZEGO KERNEL

We will need the following 3-form Dq, which is different from the usual volume 3-form in $\mathbb{R}^3$. We write $v = dx_0 \wedge dx_1 \wedge dx_2 \wedge dx_3$ for the volume form in $\mathbb{H}$. The form Dq is defined as an alternating $\mathbb{R}$ trilinear function by

(4.1) $$\langle h_1, Dq(h_2, h_3, h_4) \rangle = v(h_1, h_2, h_3, h_4)$$

for all $h_1, \ldots, h_4 \in \mathbb{H}$. The coordinate expression for Dq is

$$Dq = dx_1 \wedge dx_2 \wedge dx_3 - i_1 dx_0 \wedge dx_2 \wedge dx_3 - i_2 dx_0 \wedge dx_3 \wedge dx_1 - i_3 dx_0 \wedge dx_1 \wedge dx_2.$$

Other properties of Dq reader can fined in [20]. This is the principal form for the formulations of the integral theorems related to the q-holomorphic functions. We present the Cauchy integral theorem here.

Theorem 4.1 ([20]). *Suppose f is q-holomorphic function in an open set $U \in \mathbb{H}$. Let q_0 be a point in U, and let C be a rectifiable 3-chain which is homologous, in the singular homology of $U \setminus \{q_0\}$, to a differentiable 3-chain whose image is ∂B for some ball $B \subset U$. Then*

$$\frac{1}{2\pi^2} \int_C \frac{(q-q_0)^{-1}}{|q-q_0|^2} Dq f(q) = n f(q_0),$$

where n is the wrapping number of C about q_0.

We start from the definition of the Hardy space $\mathcal{H}^2(\mathcal{U}_1)$. In the sequel, we write $\mathbf{i} \cdot t = \sum_{m=1}^3 i_m t_m$. Let dh be Haar measure on $\mathcal{Q}$. Using the identification of $\partial \mathcal{U}_1$ with $\mathcal{Q}$ we introduce the measure $d\beta$ on $\partial \mathcal{U}_1$. So, the integration formula

$$\int_{\partial \mathcal{U}_1} F(q) d\beta(q) = \int_{\mathbb{H} \times \mathbb{R}^3} F(q_1, |q_1|^2 + \mathbf{i} \cdot t) \, dq_1 \, dt,$$

holds for a continuous function F of compact support. With this measure we can define the space $L^2(\mathcal{Q}) = L^2(\partial \mathcal{U}_1)$.

For any function F defined on $\mathcal{U}_1$, we write F_ε for its "vertical translate" (we mean that the vertical direction is given by the positive direction of $\operatorname{Re} q_2$):

$$F_\varepsilon(q) = F(q + \varepsilon \mathbf{e}), \quad \text{where} \quad \mathbf{e} = (0, 0, 0, 0, 1, 0, 0, 0).$$

If $\varepsilon > 0$, then F_ε is defined in the neighborhood of $\partial \mathcal{U}_1$. In particular, F_ε is defined on $\partial \mathcal{U}_1$.

Definition 4.1. The space $\mathcal{H}^2(\mathcal{U}_1)$ consists of all functions F holomorphic on $\mathcal{U}_1$, for which

(4.2) $$\sup_{\varepsilon > 0} \int_{\partial \mathcal{U}_1} |F_\varepsilon(q)|^2 \, d\beta(q) < \infty.$$

The norm $\|F\|_{\mathcal{H}^2(\mathcal{U}_1)}$ of F is then the square root of the left-hand side of (4.2). The space $\mathcal{H}^2(\mathcal{U}_1)$ is a Hilbert space.

We shall need the following statements concerning the holomorphic $\mathcal{H}^2$ space on the upper half space $\mathcal{U}_1$ of $\mathbb{H}^2$. We denote the upper half space $\mathcal{U}_1$ by $\mathbb{R}_+^4$. Define $\mathcal{H}^2(\mathbb{R}_+^4)$ to be a set of functions $f(q)$ which are q-holomorphic for $q = u + \mathbf{i} \cdot v$ in the upper half space $u > 0$ satisfying

(4.3) $$\left(\sup_{u > 0} \int_{\mathbb{R}^3} |f(u + \mathbf{i} \cdot v)|^2 \, dv \right)^{1/2} = \|f\|_{\mathcal{H}^2(\mathbb{R}_+^4)} < \infty.$$

The *maximal inequality*

(4.4) $$\int_{\mathbb{R}^3} \sup_{u > 0} |f(u + \mathbf{i} \cdot v)|^2 \, dv \leq A^2 \|f\|_{\mathcal{H}^2(\mathbb{R}_+^4)}^2$$

is a consequence of (4.3). This implies that there exists an $f^b \in L^2(\mathbb{R}^3)$ such that

(4.5) $f(u + \mathbf{i} \cdot v) \to f^b(v)$ as $u \to 0$,   for a. e. v,

and the identity

(4.6) $\|f^b\|_{L^2(\mathbb{R}^3)} = \|f\|_{\mathcal{H}^2(\mathbb{R}^4_+)}$.

The definition of $\mathcal{H}^2(\mathbb{R}^4_+)$ and further properties can be found in [19].

Let $q_1 \in \mathbb{H}$ and $\delta > 0$. We define f on $\mathbb{R}^4_+$ by the following $f(q_2) = F(q_1, q_2 + \delta + |q_1|^2)$.

Lemma 4.2. *Suppose that $F \in \mathcal{H}^2(\mathcal{U}_1)$ and $\delta > 0$. Then for each $q_1 \in \mathbb{H}$, the function*

$$f(q_2) = F(q_1, q_2 + \delta + |q_1|^2)$$

with $q_2 = u + \mathbf{i} \cdot v$, belongs to $\mathcal{H}^2(\mathbb{R}^4_+)$

Proof. We use the action (2.3) of group $\mathcal{Q}$ on $\mathcal{U}_1$ and others q-holomorphic automorphisms of $\mathcal{U}_1$ to simplify the statement. Let $h = [w, t] \in \mathcal{Q}$ and $q = (q_1, q_2)$. It is easy to see that $h(q + \varepsilon \mathbf{e}) = h(q) + \varepsilon \mathbf{e}$. Then $F(h(q)) \in \mathcal{H}^2(\mathcal{U}_1)$ whenever $F(q) \in \mathcal{H}^2(\mathcal{U}_1)$, since the measure $d\beta(q)$ is invariant with respect to action of h. We take $h = [q_1, 0]$ and reduce the considerations to the function $f(q_2) = F(0, q_2 + \delta)$. Applying the dilations and the property $d\beta(\delta(q)) = \delta^{10} d\beta$, we can consider $\delta = 1$ and work with

$$f(u + \mathbf{i} \cdot v) = F(0, u + 1 + \mathbf{i} \cdot v), \quad \text{for} \quad u > 0.$$

By the standard way, from the Cauchy integral Theorem 4.1 one can obtain the mean value property for q-holomorphic functions

$$(4.7) \quad |f(u + \mathbf{i} \cdot v)|^2 = |F(0, u + 1 + \mathbf{i} \cdot v)|^2 \le c \int\limits_{|q_1|^2 + |q_2|^2 < 1/4} |F(q_1, q_2 + u + 1 + \mathbf{i} \cdot v)|^2 \, dq_1 \, dq_2.$$

where $c^{-1} = \frac{\pi^4}{2^8 4!}$ is the volume of the ball of the radius $1/2$ in $\mathbb{H}^2$. Notice that since $|\operatorname{Re} q_2| < 1/2$ and $u > 0$, we have $\operatorname{Re}(q_2 + u + 1) \in (1/2, 3/2)$. Thus $\operatorname{Re}(q_2 + u + 1) > |q_1|^2$ and this guarantees that the integration in (4.7) is taken over some subset of $\mathcal{U}_1$. We write $q_2 = x + \mathbf{i} \cdot y$ and integrate (4.7) with respect to v over $\mathbb{R}^3$. After applying the Fubini theorem, we deduce

$$\int_{\mathbb{R}^3} |f(u + \mathbf{i} \cdot v)|^2 \, dv \le c \int\limits_{|q_1| \le 1/2} \int_{\mathbb{R}^3} \int\limits_{|q_2| < 1/2} |F(q_1, q_2 + u + 1 + \mathbf{i} \cdot v)|^2 \, Dq_2 \, dx \, dq_1 \, dv.$$

We take now the second integral with respect to Dq_2 and write $v = y$. We get

$$\int_{\mathbb{R}^3} |f(u + \mathbf{i} \cdot v)|^2 \, dv \le c_1 \int\limits_{|q_1| < 1/2} \int_{\mathbb{R}^3} \int\limits_{|x| < 1/2} |F(q_1, x + u + 1 + \mathbf{i} \cdot y)|^2 \, dx \, dq_1 \, dy.$$

We make the following change of variables $x + u + 1 = \varepsilon + |q_1|^2$. Since $|x| < 1/2$ and $|q_1|^2 < 1/4$, the rang of new variable ε is in the interval $(u + 1/4, u + 3/2)$. So, the last integral take the form

$$\int\limits_{u+3/2}^{u+1/4} \int\limits_{|q_1| < 1/2} \int_{\mathbb{R}^3} |F(q_1, \varepsilon + |q_1|^2 + \mathbf{i} \cdot y)|^2 \, dq_1 \, dy \, d\varepsilon$$

$$\le \frac{5}{4} \int\limits_{\mathbb{H} \times \mathbb{R}^3} |F(q_1, \varepsilon + |q_1|^2 + \mathbf{i} \cdot y)|^2 \, dq_1 \, dy = \frac{5}{4} \int\limits_{\partial \mathcal{U}_1} |F(q + \varepsilon \mathbf{e})|^2 \, d\beta(q) < \infty$$

because of (4.2). This shows that $f \in \mathcal{H}^2(\mathbb{R}^4_+)$ and we finish the proof of Lemma 4.2. $\square$

Theorem 4.2. *Suppose F belongs to $\mathcal{H}^2(\mathcal{U}_1)$. Then*

1. *There exists an $F^b \in L^2(\partial\mathcal{U}_1)$ so that $F(z + \varepsilon\mathbf{e})|_{\partial\mathcal{U}_1} \to F^b$ in the $L^2(\partial\mathcal{U}_1)$ norm, as $\varepsilon \to 0$.*
2. *The above mentioned space of F^b is a closed subspace of $L^2(\partial\mathcal{U}_1)$. Moreover*
3. *$\|F^b\|_{L^2(\partial\mathcal{U}_1)} = \|F\|_{\mathcal{H}^2(\mathcal{U}_1)}$.*

Proof. To proof Theorem 4.2 we exploit the reducing to the one dimensional case. Let us fix q_1 and consider the function $F(q) = f(q_1, q_2)$ as a function of q_2. We apply the maximal inequality (4.4) and equality (4.6) to the function $f(q_2) = F(q_1, q_2 + \delta + |q_1|^2)$. We continue to write $q_2 = x + \mathbf{i} \cdot y$. Then

$$\int_{\mathbb{R}^3} \sup_{x>0} |F(q_1, x + \delta + |q_1|^2 + \mathbf{i} \cdot y)|^2 \, dy \leq A^2 \int_{\mathbb{R}^3} |F(q_1, \delta + |q_1|^2 + \mathbf{i} \cdot y)|^2 \, dy.$$

We integrate over $q_1 \in \mathbb{H}$, write $x = \varepsilon$, use the definition of $d\beta = dq_1 \, dy$, and $|q_1|^2 = \operatorname{Re} q_2$ on the $\partial\mathcal{U}_1$. Then the last inequality can be written as

$$\int_{\partial\mathcal{U}_1} \sup_{\varepsilon>0} |F(q + \varepsilon + \delta)|^2 \, d\beta(q) \leq A^2 \int_{\partial\mathcal{U}_1} |F(q + \delta)|^2 \, d\beta(q).$$

Letting δ to 0, we see

$$\int_{\partial\mathcal{U}_1} \sup_{\varepsilon>0} |F(q + \varepsilon)|^2 \, d\beta(q) \leq A^2 \|F\|^2_{\mathcal{H}^2(\mathcal{U}_1)}.$$

We conclude from the last inequality that for almost all $q_1 \in \mathbb{H}$ the function $F(q_1, q_2 + |q_1|^2)$, as a function of q_2, is in $\mathcal{H}^2(\mathbb{R}^4_+)$ and by (4.5) we see that the limit $\lim_{\varepsilon\to 0} F(q + \varepsilon\mathbf{e}) = F^b(q)$ exists for almost every $q \in \partial\mathcal{U}_1$.

We show now the property 3. By Fatou's lemma, it follows

$$\int_{\partial\mathcal{U}_1} |F^b(q)|^2 \, d\beta(q) \leq \sup_{\varepsilon>0} \int_{\partial\mathcal{U}_1} |F(q + \varepsilon\mathbf{e})|^2 \, d\beta(q) = \|F\|^2_{\mathcal{H}^2(\mathcal{U}_1)}.$$

From the other hand, we use (4.6) and get

$$\int_{\mathbb{R}^3} |F(q_1, \varepsilon + |q_1|^2 + \mathbf{i} \cdot y))|^2 \, dy \leq \sup_{\varepsilon>0} \int_{\mathbb{R}^3} |F(q_1, \varepsilon + |q_1|^2 + \mathbf{i} \cdot y))|^2 \, dy = \int_{\mathbb{R}^3} |F(q_1, |q_1|^2 + \mathbf{i} \cdot y))|^2 \, dy.$$

Integrating over $q_1 \in \mathbb{H}$ both sides of the last inequality and taking supremum over $\varepsilon > 0$, we get the property 3.

We showed that the limit F^b in $\mathcal{H}^2(\mathcal{U}_1)$ norm exists. To complete the proof of the assertion 2 we note that the mean value property (4.7) says that for any compact $K \in \mathcal{U}_1$ there is a constant c_K, such that

$$(4.8) \qquad \sup_{q \in K} |F(q)| \leq c_K \|F\|_{\mathcal{H}^2(\mathcal{U}_1)}.$$

We conclude, that if a sequence F_n converges in $\mathcal{H}^2(\mathcal{U}_1)$ norm, then the sequence F_n converges uniformly on compact subsets of $\mathcal{U}_1$, that implies that the space $\mathcal{H}^2(\mathcal{U}_1)$ is complete with respect of its norm. The Theorem 4.2 is proved. $\qquad\square$

We determine now the Cauchy-Szegö kernel $S(q, \omega)$ for the domain $\mathcal{U}_1$. The Cauchy-Szegö kernel $S(q, \omega)$ is a quaternion valued function, defined on $\mathcal{U}_1 \times \mathcal{U}_1$ and satisfying the following conditions

1. For each $\omega \in \mathcal{U}_1$, the function $q \mapsto S(q, \omega)$ is q-holomorphic for $q \in \mathcal{U}_1$, and belongs to $\mathcal{H}^2(\mathcal{U}_1)$. This allows us to define, for each $\omega \in \mathcal{U}_1$, the boundary value $S^b(q, \omega)$ for almost all $q \in \partial\mathcal{U}_1$.

2. The kernel S is symmetric: $S(q,\omega) = \overline{S(\omega,q)}$ for each $(q,\omega) \in \mathcal{U}_1 \times \mathcal{U}_1$. The symmetry permit us to extend the definition of $S(\omega,q)$ so that for each $q \in \mathcal{U}_1$, the function $S^b(q,\omega)$ is defined for almost every $\omega \in \partial\mathcal{U}_1$.

3. The kernel S satisfies the reproducing property in the following sense

$$(4.9) \qquad F(q) = \int_{\partial\mathcal{U}_1} S(q,\omega) F^b(\omega)\, d\beta(\omega), \qquad q \in \mathcal{U}_1,$$

whenever $F \in \mathcal{H}^2(\mathcal{U}_1)$.

Theorem 4.3. *Let* $S(q,\omega) = \frac{k}{r^5(q,\omega)}$, *where*

$$(4.10) \qquad r(q,\omega) = \frac{q_2 + \bar{\omega}_2}{2} - \bar{\omega}_1 q_1 \quad and \quad k = \frac{3}{8\pi^4}.$$

Then $S(q,\omega)$ is the (unique) function that satisfies the properties $1. - 3.$ above.

Proof. We introduce the Cauchy-Szegö projection operator C. The operator C is the orthogonal projection from $L^2(\partial\mathcal{U}_1)$ to the subspace of functions $\{F^b\}$ that are boundary values of functions $F \in \mathcal{H}^2(\mathcal{U}_1)$. So for each $f \in L^2(\partial\mathcal{U}_1)$, we have that $C(f) = F^b$ for some $F \in \mathcal{H}^2(\mathcal{U}_1)$; moreover, $C(F^b) = F^b$ and C is self-adjoint: $C^* = C$.

We fix $q \in \mathcal{U}_1$. Then $(Cf)(q) = F(q)$, where F corresponds to F^b. The kernel $S(q,\omega)$ will be defined by the representation

$$(4.11) \qquad F(q) = \int_{\partial\mathcal{U}_1} S(q,\omega) f(\omega)\, d\beta(\omega).$$

The space $\mathcal{H}^2(\mathcal{U}_1)$ is identified with some subspace of $L^2(\partial\mathcal{U}_1)$. We take a basis $\{\varphi_j\}$ of this subspace. We can expect that

$$(4.12) \qquad S(q,\omega) = \sum_j \varphi_j(q)\bar{\varphi}_j(\omega).$$

(In this case we also have the symmetry property

$$\overline{S(q,\omega)} = \sum_j \overline{\varphi_j(q)\bar{\varphi}_j(\omega)} = \sum_j \varphi_j(\omega)\bar{\varphi}_j(q) = S(\omega,q).\Big)$$

Indeed, if $\sum_j |a_j|^2 < \infty$, then $\sum_j a_j\varphi_j \in \mathcal{H}^2(\mathcal{U}_1)$ and $\|\sum_j a_j\varphi_j\|_{\mathcal{H}^2(\mathcal{U}_1)} = \sum_j |a_j|^2$. Moreover, for any compact set $K \subset \mathcal{U}_1$ we have by (4.8) that

$$\sup_{q \in K} \Big| \sum_j a_j\varphi_j(q) \Big| \le c_K \Big(\sum_j |a_j|^2 \Big)^{1/2}.$$

Applying the converse of Schwart's inequality, we get

$$\Big(\sum_j |\varphi_j(q)|^2 \Big)^{1/2} \le c_K, \quad \text{for all} \quad q \in K.$$

Thus the sum (4.12) converges uniformly whenever (q,ω) belongs to a compact subset of $\mathcal{U}_1 \times \mathcal{U}_1$.

Let us take (4.12) as the definition of $S(q,\omega)$. By the symmetry $\overline{S(q,\omega)} = S(\omega,q)$ the function $\overline{S(q,\omega)}$ belongs to $\mathcal{H}^2(\mathcal{U}_1)$ for each fixed $q \in \mathcal{U}_1$. Moreover $\overline{S(q,\omega)}$ extends to $q \in \mathcal{U}_1$, $\omega \in \partial\mathcal{U}_1$, by the identity (4.12) with the series converging in the norm of $L^2(\partial\mathcal{U}_1)$.

These arguments establish the existence of the function S satisfying (4.11) and the properties 1.-3. The reproducing property (4.9) uniquely determines $\overline{S(q,\omega)}$ as an element of $\mathcal{H}^2(\mathcal{U}_1)$

for each fixed q. Together with conclusion 3 of Theorem 4.2, this shows that S is uniquely determined by the properties 1.-3.

We continue the proof establishing the precise shape of the function $S(q,\omega)$. We use the translation $h(q)$, the unitary rotation $\mathcal{R}(q)$ and the dilation $\delta(q)$ on $\mathcal{U}_1$. Observe that the measure $d\beta$ is invariant with respect to translation and unitary rotations and $d\beta(\delta(q)) = \delta^{10}d\beta$, where 10 is the homogeneous dimension of the group $\mathcal{Q}$. Notice that the space $\mathcal{H}^2(\mathcal{U}_1)$ is also takes into itself under the above mentioned transformations. Then, we have

$$F(q) = \int_{\partial\mathcal{U}_1} S(h(q), h(\omega))F^b(\omega)\, d\beta(\omega),$$

$$F(q) = \int_{\partial\mathcal{U}_1} S(\mathcal{R}(q), \mathcal{R}(\omega))F^b(\omega)\, d\beta(\omega),$$

$$F(q) = \int_{\partial\mathcal{U}_1} S(\delta(q), \delta(\omega))\delta^{10} F^b(\omega)\, d\beta(\omega).$$

We conclude

(4.13) $\quad S(q,\omega) = S(h(q), h(\omega)), \quad S(q,\omega) = S(\mathcal{R}(q), \mathcal{R}(\omega)), \quad S(q,\omega) = S(\delta(q), \delta(\omega))\delta^{10}.$

The identities (4.13) hold for each $(q,\omega) \in \mathcal{U}_1 \times \mathcal{U}_1$. They also hold for each $q \in \mathcal{U}_1$ and almost all $\omega \in \partial\mathcal{U}_1$, because for each fixed $q \in \mathcal{U}_1$ the identities (4.13) are the identities for the elements of $L^2(\partial\mathcal{U}_1)$ and, therefore, they are fulfilled for almost all $\omega \in \partial\mathcal{U}_1$.

Let $S(q) = S(q,0)$, then $S(q)$ is holomorphic on $\mathcal{U}_1$ and is independent of q_1. So, we may write $S(q) = s(q_2)$. Using the transformation of S with respect to dilation we get $s(\delta^2 q_2) = \delta^{-10}s(q_2)$ for all positive δ. We conclude that $S(q) = cq_2^{-5}$. We use now the translation $h(q)$. Let $q \in \mathcal{U}_1$ and $\omega \in \partial\mathcal{U}_1$. Then by identification of $\mathcal{U}_1$ with $\mathcal{Q}$ we have $\omega = h(0)$ and $S(q, h(0)) = S(h^{-1}(q), 0) = S(h^{-1}(q))$. We calculate the element h such that $h(0) = (\omega_1, \omega_2)$. We have

$$h = [w, t] = [\omega_1, \mathrm{Im}_1(\omega_2 - |\omega_1|^2), \mathrm{Im}_2(\omega_2 - |\omega_1|^2), \mathrm{Im}_3(\omega_2 - |\omega_1|^2)].$$

We also observe, that $\mathrm{Re}\,\omega_2 = |\omega_1|^2$. Then the second coordinate of the action h^{-1} is

$$q_2 + |\omega_1|^2 - \mathbf{i}\cdot\mathrm{Im}\,\omega_2 - 2\bar\omega_1 q_1 = q_2 + \bar\omega_2 - 2\bar\omega_1 q_1 = 2r(q,\omega)$$

by (2.3). Therefore $S(q,\omega) = kr^{-5}(q,\omega)$ with $k = 2^{-5}c$.

We calculate the constant k. Use the reproducing formula for the function $F(q) = (q_2 + 1)^{-5} = (2\overline{r(\mathbf{e}, q)})^{-5}$. Applying the reproducing formula, we get

$$2^{-5} = F(\mathbf{e}) = k\int_{\partial\mathcal{U}_1} r(e,q)^{-5}F(q)\, d\beta q$$

$$= 2^5 k\int_{\partial\mathcal{U}_1} |F(q)|^2\, d\beta q = 2^5 k\int_{q'\in\mathbb{H}}\int_{t\in\mathbb{R}^3} |F(q', |q'|^2 + \sum_{m=1}^{3} i_m t_m)|^2\, dq' dt.$$

Thus

(4.14) $$k^{-1} = 4^5 \int_{w\in\mathbb{H}} dw \int_{t\in\mathbb{R}^3} \left(|t|^2 + (|w|^2 + 1)^2\right)^{-5} dt.$$

Taking the integral over $\mathbb{R}^3$, we observe that (4.14) is reduced to the product of four values $\alpha, \beta, \gamma, \delta$, where $\alpha = 4\pi$ is the volume of the unite sphere in $\mathbb{R}^3$,

$$\beta = 2\pi^2 \quad \text{is the volume of the unite sphere in} \quad \mathbb{H},$$

$$\gamma = \int_0^\infty \frac{r^2\, dr}{(r^2 + 1)^5} = \frac{\Gamma(3/2)\Gamma(7/2)}{2\Gamma(5)}, \quad \text{and}$$

$$\delta = \int_0^\infty \frac{\rho^3 \, d\rho}{(\rho^2 + 1)^7} = \frac{\Gamma(5)}{2\Gamma(7)}.$$

Multiplying all values, we get $k = \frac{3}{8\pi^4}$. □

Notice the following.

(i) The function $r(q, \omega)$ is q-holomorphic in q and anti q-holomorphic in ω. If $q = \omega$, the function r agrees with the function (2.4).

(ii) For each fixed $\omega \in \mathcal{U}_1$ the function $r(q, \omega)$ (and hence $S(q, \omega)$) is q-holomorphic for q in a neighborhood of closure of $\mathcal{U}_1$. For each fixed $q \in \mathcal{U}_1$ the function $r(q, \omega)$ (and hence $S(q, \omega)$) is anti q-holomorphic for ω in a neighborhood of closure of $\mathcal{U}_1$. In particular, if $q \in \mathcal{U}_1$ is fixed, the boundary function $S^b(\cdot, \omega)$ is, actually, defined on all of the boundary $\partial \mathcal{U}_1$.

5. THE PROJECTION OPERATOR

We describe the projection operator C as a convolution operator on the group $\mathcal{Q}$. We remember that mapping $f \mapsto C(f)$ assigns to each element $f \in L^2(\partial \mathcal{U}_1)$ another element of $L^2(\partial \mathcal{U}_1)$ of the form $C(f) = F^b$, for some $F \in \mathcal{H}^2(\mathcal{U}_1)$. Theorem 4.2 and reproducing property (4.11) imply

$$(5.1) \qquad (Cf)(q) = \lim_{\varepsilon \to 0} F(q + \varepsilon)|_{\partial \mathcal{U}_1} = \lim_{\varepsilon \to 0} \int_{\partial \mathcal{U}_1} S(q + \varepsilon, \omega) f(\omega) \, d\beta(\omega),$$

where $q \in \partial \mathcal{U}_1$ and the limit is taken in $L^2(\partial \mathcal{U}_1)$ norm.

We now use the identification of $\partial \mathcal{U}_1$ with $\mathcal{Q}$. We write $\partial \mathcal{U}_1 \ni \omega = g(0)$ for the unique $g \in \mathcal{Q}$, $\partial \mathcal{U}_1 \ni q = h(0)$ for $h \in \mathcal{Q}$, and $d\beta(\omega) = dg$. The properties (4.13) of the Cauchy-Szegö kernel give

$$S(q + \varepsilon, \omega) = S(g^{-1}(q + \varepsilon), g^{-1}(\omega)) = S(g^{-1}(q) + \varepsilon, g^{-1}(\omega)) = S(g^{-1}(h(0)) + \varepsilon, 0).$$

We change the notation setting $K_\varepsilon(h) = S(g^{-1}(h(0)) + \varepsilon, 0)$, $f(g) = f(g(0)) = f(\omega)$, and $(Cf)(h) = (Cf)(h(0)) = (Cf)(q)$. Then (5.1) takes the form

$$(5.2) \qquad (Cf)(h) = \lim_{\varepsilon \to 0} \int_{\mathcal{Q}} K_\varepsilon(g^{-1} \circ h) f(g) \, d\beta(g),$$

for $f \in L^2(\mathcal{Q})$, where the limit is taken in $L^2(\mathcal{Q})$. We see that (5.2) is formally written as the convolution $(Cf)(h) = (K * f)(h)$, where $K = \lim_{\varepsilon \to 0} K_\varepsilon$ is a distribution.

We obtain the precise form of $K(h)$. Let $h = [w, t]$, then by identification of $\partial \mathcal{U}_1$ with $\mathcal{Q}$ and Theorem 4.3 we have $K_\varepsilon(h) = c(|w|^2 + \varepsilon + \mathbf{i} \cdot t)^{-5}$ with $c = \frac{6}{\pi^4}$. We observe that

$$K_\varepsilon(h) = \frac{-c}{2i_1 3i_2 4i_3} \frac{\partial^3}{\partial t_3 \partial t_2 \partial t_1} \Big((|w|^2 + \varepsilon + \mathbf{i} \cdot t)^{-2} \Big).$$

The function $(|w|^2 + \varepsilon + \mathbf{i} \cdot t)^{-2}$ is locally integrable on $\mathcal{Q}$ (see remark below). Passing to the limit, we see that the distribution K is given by

$$K(h) = \frac{c}{24} \frac{\partial^3}{\partial t_3 \partial t_2 \partial t_1} \Big((|w|^2 + \mathbf{i} \cdot t)^{-2} \Big)$$

and equals to the function $(|w|^2 + \mathbf{i} \cdot t)^{-5}$ away from the origin.

Remark 5.1. On the group $\mathcal{Q}$ we can introduce the homogeneous norm

$$\|h\|^2 = \|[w, t]\|^2 = |w|^2 + |t|,$$

where $|\cdot|$ denotes the Euclidean norm. The homogeneous norm is a homogeneous of order 1 with respect to dilation δ function, namely: $\|\delta(h)\| = \delta\|h\|$. We also recall the analogue of the integration formula in the polar coordinates [10]. There is a positive constant k, such that whenever f is a non-negative function on $(0, \infty)$, then

$$\int_{\mathcal{Q}} f(\|h\|)\, dh = k \int_0^\infty f(r) r^{Q-1}\, dr,$$

where Q is the homogeneous dimension of the group $\mathcal{Q}$. The homogeneous dimension of $\mathcal{Q}$ equals 10. Since the kernel of the Cauchy-Szegö projection satisfies $|K(h)| \approx \|h\|^{-10}$ then the function $(|w|^2 + \mathbf{i} \cdot t)^{-2}$ is locally integrable.

Remark 5.2. We can change the arguments that we used for the calculation of the precise form of the distribution $K(h)$. It is sufficient to differentiate only one time to obtain the integrable function. We can consider

$$K_\varepsilon(h) = \frac{-c}{4i_m} \frac{\partial}{\partial t_m} \left((|w|^2 + \varepsilon + \mathbf{i} \cdot t)^{-4} \right) \quad \text{for any} \quad m = 1, 2, 3$$

and then argue as above.

6. THE KERNEL OF THE OPERATOR Δ_λ

Recall from Section 3, the operator

$$\bar{H}(w, t) = \frac{1}{2}\left(X_0 + i_1 X_1 + i_2 X_2 + i_3 X_3\right) = \frac{\partial_l}{\partial \bar{w}} + w i_1 \partial_{t_1} + w i_2 \partial_{t_2} + w i_3 \partial_{t_3},$$

where the terms $w i_k$, $k = 1, 2, 3$, are the quaternion product. The conjugate operator is

$$H(w, t) = \frac{1}{2}\left(X_0 - i_1 X_1 - i_2 X_2 - i_3 X_3\right) = \frac{\partial_l}{\partial w} - i_1 \bar{w} \partial_{t_1} - i_2 \bar{w} \partial_{t_2} - i_3 \bar{w} \partial_{t_3}.$$

Then the commutative relation give us $[\bar{H}, H] = -2 \sum_{k=1}^3 i_k \partial_{t_k}$.

To define $\bar{\partial}_b$ operator on $\mathcal{Q}$ we take a function f and set

$$\bar{\partial}_b f = \bar{H} f\, d\bar{q},$$

where $d\bar{q} = dx_0 - \sum_{k=1}^3 i_k dx_k$. It is easy to see that

$$d\bar{q} \wedge d\bar{q} = i_1 dx_2 \wedge dx_3 + i_2 dx_3 \wedge dx_1 + i_3 dx_1 \wedge dx_2.$$

Then

$$\bar{\partial}_b^2 f = \bar{H}\bar{H} f d\bar{q} \wedge d\bar{q} \neq 0.$$

This is the difference with the Heisenberg group. We then define the formal adjoint $\bar{\partial}_b^*$ of $\bar{\partial}_b$ by

$$\bar{\partial}_b^* f = -H f \quad \text{where} \quad f = f\, d\bar{q} \quad \text{is a } (0,1)\text{-form.}$$

Finally, we can define $\square_b$ operator by

$$\square_b = \bar{\partial}_b \bar{\partial}_b^* + \bar{\partial}_b^* \bar{\partial}_b.$$

When f is a function, then

$$\bar{\partial}_b^* \bar{\partial}_b(f) = -H\bar{H}(f),$$
$$\bar{\partial}_b \bar{\partial}_b^*(f) = 0.$$

It follows that

$$-H\bar{H} = -\frac{1}{2}(H\bar{H} + \bar{H}H) + \frac{1}{2}[\bar{H}, H] = -\frac{1}{2}(H\bar{H} + \bar{H}H) - 2\sum_{k=1}^{3} i_k \partial_{t_k}$$

$$= -\frac{1}{4}\Big(\sum_{l=0}^{3} X_l^2 + 8\sum_{k=1}^{3} i_k \partial_{t_k}\Big),$$

where X_l, $l = 0, 1, 2, 3$ are defined in (3.5) and ∂_{t_k}, $k = 1, 2, 3$ are their commutators. In fact, we may define a little bit more general operator Δ_λ as follows

$$\Delta_\lambda = \sum_{l=0}^{3} X_l^2 + 4\sum_{k=1}^{3} i_k \lambda_k \partial_{t_k}.$$

The operator Δ_λ is called the subLaplacian in the literature and from a theorem of Hörmander [8], it is a subelliptic operator. In terms of coordinates on $\mathcal{Q}$, one has

$$\Delta_\lambda = \sum_{l=0}^{3} \partial_{x_l}^2 + 4|x|^2 \sum_{k=1}^{3} \partial_{t_k}^2 + 4\sum_{k=1}^{3} \big((xi_k \cdot \partial_x) + \lambda_k i_k\big)\partial_{t_k}.$$

Let us observe that Δ_λ possesses the following symmetry properties

   (i) is left invariant on $\mathcal{Q}$ with respect to the translation defined by the group multiplication from the left,
  (ii) has degree 2 with respect to the dilation δ,
 (iii) is invariant under the unitary rotation $\mathcal{R}$ on $\mathbb{R}^4$.

Recall that partial Fourier transform in the variables t_k, $k = 1, 2, 3$, is defined as follows

$$\mathcal{F}(f)(x, \tau_1, \tau_2, \tau_3) = \widetilde{f}(x, \tau_1, \tau_2, \tau_3) = \int_{\mathbb{R}^3} e^{-\sum_{k=1}^{3} i_k \tau_k t_k} f(t)\, dt,$$

$$f(x, t_1, t_2, t_3) = \frac{1}{(2\pi)^3} \int_{\mathbb{R}^3} e^{\sum_{k=1}^{3} i_k t_k \tau_k} f(\tau)\, d\tau.$$

Then

$$\mathcal{F}\Big(\frac{\partial f}{\partial t_k}\Big)(x, \tau_1, \tau_2, \tau_3) = i_k \tau_k \widetilde{f}(\tau), \qquad \mathcal{F}\Big(\frac{\partial^2 f}{\partial t_k^2}\Big)(x, \tau_1, \tau_2, \tau_3) = -\tau_k^2 \widetilde{f}(\tau).$$

Consequently,

$$\widetilde{\Delta_\lambda}(\tau) = \sum_{l=1}^{4} \partial_{x_l}^2 - 4|x|^2 \sum_{k=1}^{3} \tau_k^2 + 4\sum_{k=1}^{3} \big((xi_k \cdot \partial_x)i_k - \lambda_k\big)\tau_k.$$

We write $|\tau|^2 = \sum_{k=1}^{3} \tau_k^2$. When $\lambda = 0$, one has

$$\widetilde{\Delta_0}(\tau) = \sum_{l=1}^{4} \partial_{x_l}^2 - 4|x|^2 |\tau|^2 + 4\sum_{k=1}^{3} \big(i_k(xi_k \cdot \partial_x)\big)\partial_{t_k}.$$

The property (iii) implies that $\widetilde{\Delta_\lambda}(\tau)$ is invariant under the unitary rotation group action on $\mathbb{R}^4$. Therefore, the fundamental solution $\widetilde{K}_\lambda(x, \tau)$ of $\widetilde{\Delta_\lambda}(\tau)$ is a radial distribution. It follows that

$$\sum_{k=1}^{3} \big(xi_k \cdot \partial_x\big) i_k \tau_k \widetilde{K}_\lambda(x) = 0.$$

Hence the operator $\widetilde{\Delta_\lambda}(\tau)$ can be reduced to a Hermite operator in $\mathbb{R}^4$:

$$(6.1) \qquad \widetilde{H_\lambda}(\tau) = \sum_{l=1}^{4} \partial_{x_l}^2 - 4|x|^2 \sum_{k=1}^{3} \tau_k^2 - 4 \sum_{k=1}^{3} \lambda_k \tau_k.$$

Now by a result in [3], we know that the fundamental solution $\widetilde{K}_\lambda(x)$ of the operator $\widetilde{H_\lambda}(\tau)$ has the following form

$$\widetilde{K}_\lambda(x,\tau) = \frac{|\tau|^2}{\pi^2} \int_0^\infty e^{-4(\sum_{k=1}^3 \lambda_k \tau_k)s} \frac{e^{-|\tau||x|^2 \coth(4|\tau|s)}}{\sinh^2(4|\tau|s)} ds$$

where $|\tau|^2 = \sum_{k=1}^3 \tau_k^2$. It follows that

$$(6.2) \qquad \begin{aligned} K_\lambda(x,t) &= \frac{1}{(2\pi)^3} \int_{\mathbb{R}^3} e^{\sum_{k=1}^3 i_k t_k \tau_k} \widetilde{K}_\lambda(x,\tau) d\tau \\ &= \frac{1}{8\pi^5} \int_{\mathbb{R}^3} \int_0^\infty \frac{|\tau|^2}{\sinh^2(4|\tau|s)} e^{-|\tau||x|^2 \coth(4|\tau|s)} e^{\sum_{k=1}^3 (i_k t_k - 4\lambda_k s)\tau_k} ds d\tau. \end{aligned}$$

Let us look at these integrals more carefully. Changing variable $4|\tau|s = u$ implies that

$$\widetilde{K}_\lambda(x,\tau) = \frac{|\tau|}{4\pi^2} \int_0^\infty e^{-(\sum_{k=1}^3 \lambda_k \tau_k)\frac{u}{|\tau|}} \frac{e^{-|\tau||x|^2 \coth(u)}}{\sinh^2(u)} du.$$

Then

$$\begin{aligned} K_\lambda(x,t) &= \frac{1}{(2\pi)^3} \int_{\mathbb{R}^3} e^{\sum_{k=1}^3 i_k t_k \tau_k} \widetilde{K}_\lambda(x,\tau) d\tau \\ &= \frac{1}{(2\pi)^5} \int_{\mathbb{R}^3} e^{\sum_{k=1}^3 i_k t_k \tau_k} \int_0^\infty e^{-(\sum_{k=1}^3 \lambda_k \tau_k)\frac{u}{|\tau|}} \frac{e^{-|\tau||x|^2 \coth(u)}}{\sinh^2(u)} |\tau| \, du \, d\tau \end{aligned}$$

We introduce the polar coordinates for the τ-variable such that $\tau = |\tau|\vec{n} = |\tau|(n_1, n_2, n_3) = r\vec{n}$ where $r = |\tau|$. Then the above integral can be rewritten as

$$\begin{aligned} K_\lambda(x,t) &= \frac{1}{(2\pi)^5} \int_0^\infty \int_{S^2} e^{r(\sum_{k=1}^3 i_k t_k n_k)} \int_0^\infty e^{-(\sum_{k=1}^3 \lambda_k n_k)u - r|x|^2 \coth(u)} \frac{r^3}{\sinh^2(u)} \, du \, d\sigma \, dr \\ &= \frac{1}{(2\pi)^5} \int_{S^2} \int_0^\infty \frac{e^{-(\sum_{k=1}^3 \lambda_k n_k)u}}{\sinh^2(u)} \int_0^\infty e^{-(|x|^2 \coth(u) - \sum_{k=1}^3 i_k t_k n_k)r} r^3 \, dr \, du \, d\sigma \\ &= \frac{6}{(2\pi)^5} \int_{S^2} \int_0^\infty \frac{e^{-(\sum_{k=1}^3 \lambda_k n_k)u}}{\sinh^2(u)} \frac{du \, d\sigma}{[|x|^2 \coth(u) - \sum_{k=1}^3 i_k t_k n_k]^4}. \end{aligned}$$

Here $d\sigma$ is the surface measure on S^2. When $\lambda_k = 0$ for $k = 1, 2, 3$, the integral can be simplified. Let $v = \coth(u)$, then $dv = -\frac{du}{\sinh^2(u)}$ and the above inner integral reduces to

$$\int_0^\infty \frac{dv}{[|x|^2 v - \sum_{k=1}^3 i_k t_k n_k]^4} = -\frac{1}{3|x|^2} \frac{1}{[|x|^2 - \sum_{k=1}^3 i_k t_k n_k]^3}.$$

Hence

$$(6.3) \qquad K_0(x,t) = -\frac{2}{(2\pi)^5 |x|^2} \int_{S^2} \frac{d\sigma}{[|x|^2 - \sum_{k=1}^3 i_k t_k n_k]^3}.$$

If we put $i_1 = i$, the usual complex unity and suppose that i_2, i_3 are absent, then the group $\mathcal{Q}$ is reduced to the 2-dimensional Heisenberg group $\mathbb{C}^2 \times \mathbb{R}$ with a 4-dimensional horizontal

space and an one dimensional center. In this case the formula (6.2) reduced to the known formula obtained in [10]. Indeed, the Hermit operator (6.1) is written in the form

$$\widetilde{H}_\lambda(\tau) = \sum_{l=1}^4 \partial_{x_l}^2 - 4|x|^2\tau^2 - 4\lambda\tau.$$

Then

$$K_\lambda(x) = \frac{1}{2\pi} \int_\mathbb{R} e^{it\tau} \widetilde{K}_\lambda(x) d\tau$$

$$= \frac{1}{2\pi^3} \int_\mathbb{R} e^{it\tau} \int_0^\infty e^{-4\lambda\tau s} \frac{|\tau|^2}{\sinh^2(4|\tau|s)} e^{-|\tau||x|^2\coth(4|\tau|s)} \, ds \, d\tau$$

Changing variable $4s$ to s, the above formula becomes

$$K_\lambda(x,t) = \frac{1}{8\pi^3} \int_\mathbb{R} e^{it\tau} |\tau|^2 \int_0^\infty \frac{e^{-\lambda\tau s}}{\sinh^2(|\tau|s)} e^{-|\tau||x|^2\coth(|\tau|s)} \, ds \, d\tau$$

$$= \frac{1}{8\pi^3} \int_\mathbb{R} e^{it\tau} \tau^2 \int_0^\infty \frac{e^{-\lambda\tau s}}{\sinh^2(\tau s)} e^{-\tau|x|^2\coth(\tau s)} \, ds \, d\tau,$$

since $|\tau|^2$, $\sinh^2(|\tau|s)$, $|\tau|\coth(|\tau|s)$ are all even functions. Let $u = \tau s$, then one has

$$K_\lambda(x,t) = \frac{1}{8\pi^3} \int_\mathbb{R} e^{it\frac{u}{s}} \frac{u^2}{s^2} \int_0^\infty \frac{e^{-\lambda u}}{\sinh^2(u)} e^{-\frac{u}{s}|x|^2\coth(u)} \frac{ds}{s} \, du$$

$$= \frac{1}{8\pi^3} \int_\mathbb{R} \frac{u^2}{\sinh^2(u)} e^{-\lambda u} \int_0^\infty s^{-3} e^{-\frac{1}{s}(u|x|^2\coth(u)-itu)} \, ds \, du.$$

Once again, changing variable $\frac{1}{s}$ to s in the second integral, then we have

$$\int_0^\infty s^{-3} e^{-\frac{1}{s}(u|x|^2\coth(u)-itu)} ds = \int_0^\infty s e^{-s(u|x|^2\coth(u)-itu)} ds.$$

Now we may apply the identity

$$\frac{1}{\Gamma(m)} \int_0^\infty s^{m-1} e^{-sA} ds = \frac{1}{A^m} \qquad \text{for} \quad \text{Re}(A) > 0$$

to the above integral, then we have

$$K_\lambda(x,t) = \frac{\Gamma(2)}{8\pi^3} \int_{-\infty}^{+\infty} \frac{e^{-\lambda u}}{\sinh^2(u)} \frac{du}{(|x|^2\coth(u) - it)^2}.$$

Denote

$$r = (|x|^4 + t^2)^{\frac{1}{4}} \quad \text{and} \quad e^{-i\phi} = r^{-2}(|x|^2 - it)$$

with $\phi \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. Using the identity

$$\cosh(u + i\phi) = \cosh(u)\cos\phi + i\sinh(u)\sin\phi,$$

one has

(6.4) $$K_\lambda(x,t) = \frac{1}{8\pi^3} \int_{-\infty}^\infty \frac{e^{-\lambda u}}{[r^2\cosh(u + i\phi)]^2} du.$$

Changing the contour, the formula (6.4) becomes

$$K_\lambda(x,t) = \frac{1}{8\pi^3 r^4} e^{i\lambda\phi} \int_{-\infty}^\infty \frac{e^{-\lambda u}}{[\cosh(u)]^2} du.$$

The above integral can be evaluated as follows:

$$K_\lambda(x,t) = \frac{1}{4\pi^3} r^{-4} e^{i\lambda\phi} \Gamma\left(\frac{2+\lambda}{2}\right) \Gamma\left(\frac{2-\lambda}{2}\right)$$

$$= \frac{1}{4\pi^3} \Gamma\left(\frac{2+\lambda}{2}\right) \Gamma\left(\frac{2-\lambda}{2}\right) (|x|^2 - it)^{-\frac{2+\lambda}{2}} (|x|^2 + it)^{-\frac{2-\lambda}{2}}.$$

From the above formula, we know that the kernel can be extended from $|\operatorname{Re}(\lambda)| < 2$ to the region $\mathbb{C} \setminus \Lambda$ where

$$\Lambda = \{\pm(2 + 2k) : \ k \in \mathbb{Z}_+\}.$$

This coincides with the result obtained by Folland and Stein [10]. In particular, when $\lambda = 0$, one has

$$K_0(x,t) = \frac{1}{4\pi^3} (|x|^2 + it)^{-\frac{2}{2}} (|x|^2 - it)^{-\frac{2}{2}} = \frac{1}{4\pi^3} (|x|^4 + t^2)^{-1}.$$

which recovered a result of Folland [9].

Remark 6.1. Let us look on Cauchy-Riemann equations on a domain $D \subset \mathbb{H}^2$. Let $u(q_1, q_2)$ be a smooth function of two quaternion variables, defined in a domain $D \subset \mathbb{H}^2$. Then the Cauchy-Riemann equations is expressed by the system

(6.5)
$$\bar{\partial}_{q_1} u = \frac{\partial u}{\partial \bar{q}_1} = f_1$$

$$\bar{\partial}_{q_2} u = \frac{\partial u}{\partial \bar{q}_2} = f_2.$$

Since the product $\bar{\partial}_{q_1} \bar{\partial}_{q_2}$ is not commutative, we can not obtain the compatibility condition on f of the type

$$\frac{\partial f_2}{\partial \bar{q}_1} = \frac{\partial f_1}{\partial \bar{q}_2}.$$

So the result $\bar{\partial}_b^2 \neq 0$ is quite different from the Heisenberg group. Moreover, calculations of forms for quaternion is more complicated. The simplest anticommutative property

(6.6)
$$dq_1 \wedge dq_2 = -dq_2 \wedge dq_1$$

does not hold. This implies that $dq \wedge dq \neq 0$. So even if $[\bar{H}, \bar{H}] = 0$ we can not expect $\bar{\partial}_b^2$, since (6.6). It was proved in [1] that the system (6.5) has a C^∞ solution $u(q_1, q_2)$ defined in a convex domain $D \subset \mathbb{H}^2$ if and only if the vector (f_1, f_2) solve a matrix the second order differential equation that can be considered as an analogue of the compatibility condition. However, can we find an elegant way to describe is the operator $\Box_b = \bar{\partial}_b \bar{\partial}_b^* + \bar{\partial}_b^* \bar{\partial}_b$? Furthermore, what is the correct setting for the $\bar{\partial}$-Neumann problem on a bounded domain in $\mathbb{H}^2$?

Remark 6.2. How to define correctly the pseudoconvexity in quaternion setting? We need the analogue of Hermitian matrix and Levi matrix. Even for the case of 1-dimension it is need to understand. We have the tangential Cauchy-Riemann-Fueter operator $\bar{H}$, such that $[H, \bar{H}] = 2 \sum_{k=1}^3 i_k \partial_{t_k}$. We can say that $\sum_{k=1}^3 i_k \partial_{t_k}$ is an elementary pure imaginary vector field defining the missing directions of real dimension 3. Then the Levi matrix is the number 2, which is positive definite. What is in general case?

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Department of Mathematics, Georgetown University, Washington D.C. 20057, USA
E-mail address: chang@georgetown.edu

Department of Mathematics, University of Bergen, Johannes Brunsgate 12, Bergen 5008, Norway
E-mail address: irina.markina@uib.no